THE MOTION OF CONTROLLED MECHANICAL SYSTEMS WITH PRESCRIBED CONSTRAINTS (SERVOCONSTRAINTS)

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The problem of motion of controlled mechanical systems under prescribed constraints (servoconstraints) is considered. The problem consists in applying a control to a given system to assure that in the course of its motion a given number of specified relationships among the variables of the problem (prescribed system constraints) are fulfilled. This involves finding the required control and determining the character of motion if the prescribed constraints are fulfilled exactly.

In a sense, the problem is not new. In 1922 Béguin published his thesis on gyroscopic compasses in which he pointed out the existence of a special class of mechanical systems which were then called servosystems and which now might more properly be termed controlled mechanical systems. The distinctive feature of these systems lies in the special means of effecting certain limitations (called "servoconstraints" by Appell) on the system motion. In his work Appell [1] developed the theory of such mechanical systems. However, consideration of servosystems as controlled systems makes possible a new approach to the problem, enabling one to pose the problem of motion of systems under prescribed constraints (servoconstraints) more precisely and to obtain more complete results.

In the present paper we shall develop a general formal method for solving the problem. The method consists in reducing the prescribed constraints of the system to real constraints and in adding to the equations of system motion obtained under this assumption the dynamic conditions of fulfillment of the prescribed constraints (the conditions of equality to zero of their reactions). The resulting equations constitute the solution of the problem.

1. A controlled system of n material points moves in some stationary Cartesian coordinate system. We denote by $x_1, x_2, x_3, m_1 = m_2 = m_3$ the coordinates and mass of the first point of the system, by $x_4, x_5, x_6, m_4 = m_5 = m_6$ the coordinates and mass of the second point, etc. The points of the system are subject to ideal holonomic and nonholonomic constraints which may include constraints dependent on the control parameters [2]. Let the equations of the system be

$$f_\rho(t, x_1, \ldots, x_{3n}, u_1, \ldots, u_k) = 0$$

$$f_{g+\varkappa}(t, x_1, \ldots, x_{3n}, x_1', \ldots, x_{3n}', u_1, \ldots, u_k) = 0 \qquad (1.1)$$

where $u_1, \ldots, u_k$ are the system control parameters.

The forces acting on the system will be assumed to be specific functions of time, the coordinates and velocities of the system points, and the control parameters. By X_1, X_2, X_3 we denote the components of the coordinate axes of the resultant of the active forces acting on the first point of the system, and by X_4, X_5, X_6 the components of the resultant of the forces acting on the second point, etc.

As in the case of uncontrolled systems, the motion of controlled mechanical systems is described by the fundamental equation of mechanics [2]

$$\sum (m_i x_i'' - X_i)\, \delta x_i = 0 \tag{1.2}$$

where $\delta x_1, \ldots, \delta x_{3n}$ are the components of the possible displacements of the system given by the usual relations

$$\sum \frac{\partial f_\rho}{\partial x_i}\, \delta x_i = 0, \qquad \sum \frac{\partial f_{g+\varkappa}}{\partial x_i'}\, \delta x_i = 0$$

From Eq. (1.2) we obtain the following general equations of motion of a controlled mechanical system:

$$m_i x_i'' = X_i + \sum \lambda_\rho \frac{\partial f_\rho}{\partial x_i} + \sum \lambda_{g+\varkappa} \frac{\partial f_{g+\varkappa}}{\partial x_i'} \tag{1.3}$$

where λ_ρ and $\lambda_{g+\varkappa}$ are indefinite Lagrange multipliers. Summation in Eqs. (1.3) is carried out over all constraints (1.1) of the mechanical system under consideration.

Let us consider the following problem. We are given a certain number of relations

$$\begin{aligned} &\varphi_\sigma (t, x_1, \ldots x_{3n}, u_1, \ldots, u_k) = 0 \\ &\varphi_{h+\pi} (t, x_1, \ldots, x_{3n}, x_1', \ldots, x_{3n}', u_1, \ldots, u_k) = 0 \end{aligned} \tag{1.4}$$

By effecting appropriate control of the system we must guarantee exact fulfillment of these relations in the course of system motion. A more precise way of stating the problem is to ask the following questions: Once relations (1.4) are fulfilled at the initial instant, wh is the control required to guarantee their fulfillment in the future? What will be the consequent motion of the system once such a control is applied?

Relations (1.4) will be called the prescribed constraints or servoconstraints of the system. These constraints are, so to speak, "enforced" by the appropriate system control. In this sense prescribed constraints are identical to the servoconstraints of Béguin and Appell. This is clearly apparent from the following description of servoconstraints given by Appell [1]:

" There exists an important class of mechanisms in which constraints are realized by a means entirely different from those considered up to now. For mechanisms of this class the means of realization of the constraints cannot be ignored.

" The constraints realized by these mechanisms are arbitrary; most often they are holonomic. But the constraints are effected not by simple contact – not, so to speak, passively. Their realization involves the use of various forces (electromagnetic forces, compressed air pressure, etc.), i.e. of auxiliary energy sources which are automatically actuated and automatically controlled in such a way that the given constraint is realized at each instant. Such a mechanism can be compared with a living organism acting by direct contact and regulating its efforts in such a way as to fulfill a given constraint."

Appell did not use the term "control" in his description of servoconstraints. This term simply did not exist at the time. But it is clear that in describing the means of fulfillment of servoconstraints Appell had in mind that which we now call "control".

The above problem of the motion of a controlled mechanical system with prescribed constraints can be solved by simultaneous investigation of Eqs. (1.3), (1.1), and (1.4). The required system control is then obtained from the compatibility condition for these equations. The motion can then be determined by direct integration of Eqs. of system motion (1.3).

This method of solving the problem is hardly expedient, since it involves integration of a system of differential equations of, generally speaking, a rather high order. This necessarily gives rise to the problem of lowering the order of the differential equations of the problem with due regard for its specific features. In particular, there arises the task of deriving specialized equations for the case under investigation which describe the problem fully but at the same time are of the lowest possible order.

The classical method of solving this problem for ordinary mechanical systems is to introduce generalized coordinates. As will be shown below, the problem can be solved in exactly the same way for the motion of controlled mechanical systems with prescribed constraints.

2. Let us assume that equations independent in the control parameters have been isolated from Eqs. (1.1), and that the control parameters have been eliminated from the remaining Eqs. (1.1). Further, we assume that the transformed Eqs. (1.1) are numbered in such a way that the parametric equations form the first a equations in the group of holonomic constraint equations and the first b equations in the group of nonholonomic constraint equations.

Taking into account the equations of the holonomic constraints independent of the control parameters, we introduce the generalized parameters of the system $q_1, \ldots, q_s$. Then, taking account of the equations of the nonholonomic constraints independent of the control parameters, we introduce the generalized velocity parameters $\omega_1, \ldots, \omega_p$.

Note 2.1. The exclusive use of constraint equations independent of the control parameters in introducing the generalized coordinates is intended to exclude derivatives of the control parameters from the equations of motion of the system, and thus to avoid artificial increases in the order of the final system of differential equations of the problem.

Expressing the Cartesian coordinates of the system in terms of its generalized coordinates and the derivatives of the latter with respect to time in terms of the generalized velocity parameters, we have

$$x_i = x_i(t, q_1, \ldots, q_s), \qquad q_\alpha' = A_\alpha(t, q_1, \ldots, q_s, \omega_1, \ldots, \omega_p) \tag{2.1}$$

These equations enable us to transform the general equations of system motion (1.3) to the form

$$\frac{\partial S}{\partial \omega_\xi'} = \Omega_\xi + \sum_{\rho=1}^{\alpha} \lambda_\rho \frac{\partial f_\rho'}{\partial \omega_\xi} + \sum_{\varkappa=1}^{\beta} \lambda_{g+\varkappa} \frac{\partial f_{g+\varkappa}}{\partial \omega_\xi} \tag{2.2}$$

where S is the energy of accelerations of the system and Ω_ξ are the generalized forces of the system referred to the velocity parameters. The latter are of the form

$$\Omega_\xi = \sum X_i \frac{\partial x_i}{\partial q_\alpha} \frac{\partial A_\alpha}{\partial \omega_\xi} = \sum X_i \frac{\partial x_i'}{\partial \omega_\xi}$$

Eqs. (2.2) are ordinary Appell equations in which the (real) parametric constraints of the system are taken into account by way of indefinite multipliers. To Eqs. (2.2) we must add the equations of the real parametric constraints of the system after we have written these in terms of the generalized coordinates and velocity parameters

$$f_\rho(t, q_1, \ldots, q_s, u_1, \ldots, u_k) = 0 \tag{2.3}$$

$$f_{g+\varkappa}(t, q_1, \ldots, q_s, \omega_1, \ldots, \omega_p, u_1, \ldots, u_k) = 0 \quad (\rho = 1, 2, \ldots, a; \quad \varkappa = 1, 2, \ldots, b)$$

as well as Eqs. (1.4) of the prescribed constraints of the system expressed in the same variables.

3. Consideration of the real constraints of the system independent of the control parameters by way of generalized coordinates and velocity parameters enables us to exclude a number of "excess" (dependent) variables from the differential equations of the problem and thus to reduce their overall order. Further elimination of dependent variables from the differential equations and therefore further reduction of the overall order of the equations can be effected through consideration of the prescribed constraints of the system.

Let us suppose that, as in the case of Eqs. (1.1), we have isolated from Eqs. (1.4) of the prescribed constraints of the system the equations independent in the control parameters both among themselves and in relation to the parametric equations of transformed system (1.1); let us assume, moreover, that the control parameters have been eliminated from the remaining Eqs. (1.4). Next, we renumber the transformed Eqs. (1.4) in such a way that only the first c equations in the group of holonomic prescribed constraint equations and the first d equations in the group of nonholonomic prescribed constraint equations depend on the control parameters, We assume that the generalized coordinates and velocity parameters of the system have been chosen in such a way that the fulfillment of the prescribed constraints independent of the control parameters is equivalent to the vanishing of the last $s - r$ generalized coordinates and the last $p - q$ generalized velocity parameters.

For convenience, we write

$$q_{r+1} = \eta_1, \ldots, \quad q_s = \eta_{s-r}, \quad \omega_{q+1} = \pi_1, \ldots, \quad \omega_p = \pi_{p-q} \tag{3.1}$$

The equations of the prescribed constraints of the system independent of the control parameters can then be written as

$$\eta_1 = 0, \ldots, \quad \eta_{s-r} = 0, \quad \pi_1 = 0, \ldots, \quad \pi_{p-q} = 0 \tag{3.2}$$

Taking account of the latter equations, we can rewrite the equations of the parametric prescribed constraints as

$$\varphi_\sigma(t, q_1, \ldots, q_r, u_1, \ldots, u_k) = 0 \tag{3.3}$$

$$\varphi_{h+\pi}(t, q_1, \ldots, q_r, \omega_1, \ldots, \omega_q, u_1, \ldots u_k) = 0 \qquad (\sigma = 1, 2, \ldots, c; \ \pi = 1, 2, \ldots, d)$$

Now let us substitute conditions (3.2) into Eqs. (2.3) and (2.2). We first rewrite both of the latter with allowance for notation (3.1). Eqs. (2.3) then become

$$f_\rho(t, q_1, \ldots, q_r, \eta_1, \ldots, \eta_{s-r}, u_1, \ldots, u_k) = 0$$

$$f_{g+\varkappa}(t, q_1, \ldots, q_r, \eta_1, \ldots, \eta_{s-r}, \omega_1, \ldots, \omega_q, \pi_1, \ldots, \pi_{p-q}, u_1, \ldots, u_k) = 0$$

$$(\rho = 1, 2, \ldots, a; \ \varkappa = 1, 2, \ldots, b)$$

Eqs. (2.2) become

$$\frac{\partial S}{\partial \omega_\xi'} = \Omega_\xi + \sum_{\rho=1}^{a} \lambda_\rho \frac{\partial f_\rho'}{\partial \omega_\xi} + \sum_{\varkappa=1}^{b} \lambda_{g+\varkappa} \frac{\partial f_{g+\varkappa}}{\partial \omega_\xi},$$

$$\frac{\partial S}{\partial \pi_\tau'} = \Pi_\tau + \sum_{\rho=1}^{a} \lambda_\rho \frac{\partial f_\rho'}{\partial \pi_\tau} + \sum_{\varkappa=1}^{b} \lambda_{g+\varkappa} \frac{\partial f_{g+\varkappa}}{\partial \pi_\tau} \qquad (\xi = 1, 2, \ldots, q; \quad \tau = 1, 2, \ldots, p-q) \tag{3.4}$$

where in the second group of equations we use the notation

$$\Pi_\tau = \Omega_{q+\tau} \qquad (\tau = 1, 2, \ldots, p-q)$$

For convenience, we denote the result of substituting conditions (3.2) into any given expression A by A^*. Eqs. (2.3) into which we have substituted conditions (3.2) can then be written as

$$f_\rho^*(t, q_1, \ldots, q_r, u_1, \ldots, u_k) = 0, \quad \overset{*}{f}_{g+\varkappa}(t, q_1, \ldots, q_r, \omega_1, \ldots, \omega_q, u_1, \ldots, u_k) = 0$$

$$(\rho = 1, 2, \ldots, a; \ \varkappa = 1, 2, \ldots, b) \tag{3.5}$$

Substituting conditions (3.2) into Eqs. (3.4), we take into account the identities

$$\left(\frac{\partial A}{\partial \omega'}\right)^* = \frac{\partial A^*}{\partial \omega'}, \qquad \left(\frac{\partial A}{\partial \omega}\right)^* = \frac{\partial A^*}{\partial \omega}, \qquad \left(\frac{dA}{dt}\right)^* = \frac{dA^*}{dt}$$

whose validity can be readily demonstrated.

Making use of these identities, we can substitute conditions (3.2) into Eqs. (3.4) and then reduce them to

$$\frac{\partial S^*}{\partial \omega_\xi'} = \Omega_\xi^* + \sum_{\rho=1}^{a} \lambda_\rho \frac{\partial f_\rho^{*\prime}}{\partial \omega_\xi} + \sum_{\varkappa=1}^{b} \lambda_{g+\varkappa} \frac{\partial \overset{*}{f}_{g+\varkappa}}{\partial \omega_\xi} \tag{3.6}$$

$$\left(\frac{\partial S}{\partial \pi_\tau'}\right)^* = \Pi_\tau^* + \sum_{\rho=1}^{a} \lambda_\rho \left(\frac{\partial f_\rho'}{\partial \pi_\tau}\right)^* + \sum_{\varkappa=1}^{b} \lambda_{g+\varkappa} \left(\frac{\partial f_{g+\varkappa}}{\partial \pi_\tau}\right)^* \tag{3.7}$$

It remains for us now to substitute conditions (3.2) into group of Eqs. (2.1) relating the derivatives of the generalized coordinates to the generalized velocity parameters. To this end we rewrite these equations taking account of notation (3.1). We have

$$q_\lambda' = A_\lambda (t, q_1, \ldots, q_r, \eta_1, \ldots, \eta_{s-r}, \omega_1, \ldots, \omega_q, \pi_1, \ldots, \pi_{p-q})$$
$$\eta_\gamma' = A_{r+\gamma} (t, q_1, \ldots, q_r, \eta_1, \ldots, \eta_{s-r}, \omega_1, \ldots, \omega_q, \pi_1, \ldots, \pi_{p-q})$$

Introducing conditions (3.2) into these equations, we obtain

$$\begin{aligned} q_\lambda' &= A_\lambda^* (t, q_1, \ldots, q_r, \omega_1, \ldots, \omega_q) \\ 0 &= A_{r+\gamma}^* (t, q_1, \ldots, q_r, \omega_1, \ldots, \omega_q) \end{aligned} \tag{3.8}$$

By hypothesis, the variables $q_1, \ldots, q_r, \omega_1, \ldots, \omega_q$ must remain independent upon fulfillment of the prescribed constraints of the system. Hence, the equations of the second group of system (3.8) must be satisfied identically. The remaining Eqs. (3.8) together with Eqs. (3.6) and (3.7) and the equations of the parametric real (3.5) and prescribed (3.3) constraints of the system form the complete system of equations of the problem of motion of a controlled mechanical system with prescribed constraints under consideration.

4. We shall now explain the mechanical significance of Eqs. (3.6) and (3.7).

It is readily apparent that Eqs. (3.6) constitute the equations of motion of the mechanical system under consideration provided its prescribed constraints which are independent of the control parameters are interpreted as real constraints and are taken into account accordingly.

Eqs. (3.7) mean that the reactions of the prescribed constraints of the system, insofar as the latter are interpreted as real constraints, must be equal to zero in the course of system motion. This implies that the fulfillment of the prescribed constraints is achieved exclusively by way of the external active forces and reactions of the real parametric constraints acting on the system.

Note 4.1. The term "(generalized) constraint reaction" used above denotes the totality of increments to be added to the generalized forces of the system in order to assure continuance of its initial motion. This definition of constraint reaction is in complete agreement with the general mechanical significance of the term and is, in fact, identical to the definition of a generalized constraint reaction given in [3].

Thus, a mechanical system obeys prescribed constraints as though these constraints were real; the reactions of the latter must equal zero, however.

We shall call Eqs. (3.6) the equations of motion of a mechanical system with prescribed constraints. Eqs. (3.7) will be called the conditions of equality to zero of the prescribed constraint reactions, or (since they are ultimately the conditions imposed on the forces in order to guarantee fulfillment of the prescribed constraints) the dynamic conditions of fulfillment of the prescribed system constraints. The possibility of interpreting and allowing for the prescribed constraints of a mechanical system as real constraints will be called the prescribed-to-real constraint reduction principle.

We note that the principle of reduction to real constraints applies both to prescribed constraints independent of the control parameters and to parametric prescribed constraints.

In fact, in taking account of both parametric and real constraints one must introduce terms corresponding to these constraints into the equations of the problem by way of indefinite Lagrange multipliers. But these multipliers are equal to zero because they are proportional to the reactions of the corresponding constraints, and the latter must equal zero by virtue of the reduction principle. For this reason, all terms associated with parametric prescribed constraints drop out of the equations of the problem. The equations of parametric prescribed constraints are therefore simply added on to the equations of the problem, precisely as we have been doing from the very start.

5. The chief practical value of the reduction principle lies in the possibility it affords of deriving immediately the equations of motion of a controlled mechanical system with prescribed constraints.

To this end, the parametric constraints (first the real and then the parametric constraints independent of the latter and of each other) are isolated from the totality of (real and prescribed) system constraints. The control parameters are eliminated from the remaining constraint equations.

The equations of motion of the system with consideration of all its constraints are then constructed in accordance with the reduction principle. It is suggested that the parametric constraints of the system be allowed for by means of indefinite multipliers, and the constraints independent of the control parameters by the introduction of generalized coordinates and velocity parameters. This method of allowing for constraints affords a maximum reduction in the order of the differential equations of the problem.

To the equations of system motion must be added the dynamic conditions of fulfillment of the prescribed constraints, the equations of the parametric real and prescribed constraints, and the equations relating the derivatives of the generalized coordinates to the velocity parameters.

In deriving the dynamic conditions, once its equations of motion have been constructed the system is liberated of prescribed constraints, while the totality of its generalized coordinates and velocity parameters is supplemented by the required number of new generalized coordinates and velocity parameters. The dynamic conditions of fulfillment of the prescribed constraints are obtained through formal consideration of the prescribed constraint equations in the equations of motion of the liberated system corresponding to the additional generalized coordinates and velocity parameters.

We derived the dynamic conditions in the form of Eqs. (3.7). It is true that in deriving them we made use of a special set of additional generalized coordinates and velocity parameters for which the equations of the prescribed constraints (independent of the control parameters) could be written in the simplest form (3.2). However, the general scheme for constructing dynamic conditions presented above makes it possible to obtain equations equivalent to (3.7) for practically any set of additional generalized coordinates and velocity parameters of the system.

In fact, let us assume that upon liberation of the system from prescribed constraints the additional generalized coordinates and velocity parameters are chosen arbitrarily. We denote these by $\zeta_1,\ldots,\zeta_{s-r},\sigma_1,\ldots,\sigma_{p-q}$, respectively. Any two sets of generalized coordinates and velocity parameters of the system can stand in a one-to-one correspondence. Such a correspondence likewise exists between the arbitrary and previously introduced special sets of the indicated variables. Let us assume that this correspondence is expressed by Eqs.

$$\begin{aligned} \zeta_\varepsilon &= \zeta_\varepsilon(t,\ q_1\ldots,q_r,\ \eta_1,\ldots,\eta_{s-r}) \\ \sigma_\nu &= \sigma_\nu(t,\ q_1,\ldots,\ q_r,\ \eta_1,\ldots,\ \eta_{s-r},\ \omega_1,\ldots,\omega_q,\pi_1,\ldots,\ \pi_{p-q}) \end{aligned} \tag{5.1}$$

Since $q_1,\ldots,q_r,\omega_1,\ldots,\omega_q$ enter into both sets of generalized coordinates and velocity parameters, by virtue of the one-to-one character of the correspondence expressed by Eqs. (5.1) the latter must be solvable for the variables $\eta_1,\ldots,\eta_{s-r},\pi_1,\ldots,\pi_{p-q}$. This means, generally speaking, that the inequalities

$$\frac{\partial(\zeta_1,\ldots,\zeta_{s-r})}{\partial(\eta_1,\ldots,\eta_{s-r})}\neq 0,\qquad \frac{\partial(\sigma_1,\ldots,\sigma_{p-q})}{\partial(\pi_1,\ldots,\pi_{p-q})}\neq 0 \tag{5.2}$$

must be valid.

Taking into account Eqs. (5.1), we obtain

$$\frac{\partial S}{\partial \pi'_\tau}=\sum\frac{\partial S}{\partial \sigma_\nu'}\frac{\partial \sigma_\nu}{\partial \pi_\tau},\ \Pi_\tau=\sum H_\nu\frac{\partial \sigma_\nu}{\partial \pi_\tau}\ \left(\mathrm{H}_\nu=\sum X_i\frac{\partial x_i'}{\partial \sigma_\nu}\right)$$

$$\frac{\partial f_\rho'}{\partial \pi_\tau}=\sum\frac{\partial f_\rho'}{\partial \sigma_\nu}\frac{\partial \sigma_\nu}{\partial \pi_\tau},\qquad \frac{\partial f_{g+\varkappa}}{\partial \pi_\tau}=\sum\frac{\partial f_{g+\varkappa}}{\partial \sigma_\nu}\frac{\partial \sigma_\nu}{\partial \pi_\tau}$$

and then

$$\frac{\partial S}{\partial \pi_\tau'}-\Pi_\tau-\sum_{\rho=1}^{a}\lambda_\rho\frac{\partial f_\rho'}{\partial \pi_\tau}-\sum_{\varkappa=1}^{b}\lambda_{g+\varkappa}\frac{\partial f_{g+\varkappa}}{\partial \pi_\tau}=$$

$$=\sum\left(\frac{\partial S}{\partial \sigma_\nu'}-\mathrm{H}_\nu-\sum_{\rho=1}^{a}\lambda_\rho\frac{\partial f_\rho'}{\partial \sigma_\nu}-\sum_{\varkappa=1}^{b}\lambda_{g+\varkappa}\frac{\partial f_{g+\varkappa}}{\partial \sigma_\nu}\right)\frac{\partial \sigma_\nu}{\partial \pi_\tau}$$

By virtue of these identities the second inequality of (5.2) implies that fulfillment of conditions (3.7) is equivalent to the fulfillment of the conditions

$$\left(\frac{\partial S}{\partial \sigma_\nu'}-\mathrm{H}_\nu-\sum_{\rho=1}^{a}\lambda_\rho\frac{\partial f_\rho'}{\partial \sigma_\nu}-\sum_{\varkappa=1}^{b}\lambda_{g+\varkappa}\frac{\partial f_{g+\varkappa}}{\partial \sigma_\nu}\right)^*=0 \tag{5.3}$$

The substitution of identities (3.2) into Eqs. (3.7) is clearly analogous to the substitution into (5.3) of the Eqs.

$$\zeta_\varepsilon=\zeta_\varepsilon^*(t,q_1,\ldots,q_r),\qquad \sigma_\nu=\sigma_\nu^*(t,\ q_1,\ldots,\ q_r,\ \omega_1,\ldots,\omega_q) \tag{5.4}$$

obtained from Eqs. (5.1) by substituting into them conditions (3.2). This, incidentally, implies that Eqs. (5.4) are the equations of prescribed constraints (independent of the control parameters) written with the aid of the additional generalized coordinates ζ and velocity parameters σ. Eqs. (5.4) can be obtained directly without prior establishment of Eqs. (5.1).

Eqs. (5.3) are the dynamic conditions of fulfillment of the prescribed constraints written out for the arbitrarily chosen additional generalized coordinates and velocity parameters. From their form we see that the method used for their construction is in complete accord with the scheme described above.

6. An important special case of the general mechanical systems considered above are holonomic systems subject to holonomic prescribed constraints.

Analytically this special case is characterized by the fact that there are no nonholonomic equations among constraint Eqs. (1.1) and (1.4). It corresponds to the equations of motion and general dynamic conditions obtainable from (3.6) and (5.3) by the elimination of terms corresponding to nonholonomic constraint equations. Thus, the equations of motion of a holonomic mechanical system subject to holonomic prescribed constraints can be written as

$$\frac{\partial S^*}{\partial \omega_\xi'}=\Omega_\xi^*+\sum_{\rho=1}^{a}\lambda_\rho\frac{\partial f_\rho^{*\prime}}{\partial \omega_\xi'} \tag{6.1}$$

The dynamic conditions of fulfillment of the prescribed constraints take the form

$$\left(\frac{\partial S}{\partial \sigma_\nu'}-\Pi_\nu-\sum_{\rho=1}^{a}\lambda_\rho\frac{\partial f_\rho'}{\partial \sigma_\nu}\right)^*=0 \tag{6.2}$$

Here, as in Eqs. (5.3), the superscript * means that Eqs. (5.4) must be taken into account. However, the prescribed constraint equations consist solely of the first group of Eqs. (5.4). The second group of these equations is formal and reflects the transition from the velocity parameters ω to the parameters σ in the derivation of the dynamic conditions.

Eqs. (6.1) and (6.2) conform to the expressions for the general equations of motion and dynamic conditions containing velocity parameters. If derivatives of the generalized coordinates are used as the velocity parameters, i.e. if one sets $\omega_\xi = q_\xi'$, $\sigma_\nu = \zeta_\nu'$, then Eqs. (6.1) and (6.2) can be reduced to a form more typical of holonomic systems.

In fact, in this case

$$\frac{\partial S^*}{\partial \omega_\xi'} = \frac{\partial S^*}{\partial q_\xi''} = \frac{d}{dt}\frac{\partial T^*}{\partial q_\xi'} - \frac{\partial T^*}{\partial q_\xi}, \qquad \frac{\partial f_\rho^{*\prime}}{\partial \omega_\xi} = \frac{\partial f_\rho^{*\prime}}{\partial q_\xi'} = \frac{\partial f_\rho^*}{\partial q_\xi}$$

where T^* is the kinetic energy of the system constructed with allowance for all of its constraints (both real and prescribed), so that Eqs. (6.1) can be written as

$$\frac{d}{dt}\frac{\partial T^*}{\partial q_\xi'} - \frac{\partial T^*}{\partial q_\xi} = \Omega_\xi^* + \sum_{\rho=1}^{a} \lambda_\rho \frac{\partial f_\rho^*}{\partial q_\xi} \tag{6.3}$$

In precisely the same way dynamic conditions (6.2) can be reduced to

$$\left(\frac{d}{dt}\frac{\partial T}{\partial \zeta_\nu'} - \frac{\partial T}{\partial \zeta_\nu} - H_\nu - \sum_{\rho=1}^{a} \lambda_\rho \frac{\partial f_\rho}{\partial \zeta_\nu}\right)^* = 0 \tag{6.4}$$

Here T is the kinetic energy of the system liberated from prescribed constraints. In Eqs. (5.4) which must be allowed for in (6.4), the second group of equations is now a consequence of the first and must be discarded.

7. Examples. Holonomic systems.

a) A pursuit problem. The material point M_2 pursues another material point M_1, approaching it according to a parallel homing procedure. We are to find the conditions imposed on the forces under which approach is possible and to describe the motion of the material points.

Parallel homing requires matching of the motion of the point M_2 to that of the point M_1 in such a way that the direction of the straight line M_2M_1 remains constant in space. Let us take the coordinate system with its Ox axis parallel to this direction. We denote by x_1, y_1, z_1 and x_2, y_2, z_2 the coordinates of the first and second material points in this coordinate system. The parallel homing requirement then consists in the fulfillment of the prescribed constraints

$$y_2 - y_1 = 0, \qquad z_2 - z_1 = 0 \tag{7.1}$$

during the motion of the system.

Let us find the equations of motion of the points M_1 and M_2 and the dynamic conditions of fulfillment of prescribed constraints (7.1). Taking account of (7.1), we conclude that our system of two material points has four degrees of freedom. As the generalized coordinates of the system we take the coordinates of the point M_1 and the coordinate x_2 of the point M_2.

Constructing the Lagrange equations, we obtain the equations of motion of the system (Eq. (3.7)). These are of the form

$$\begin{gathered} m_1x_1'' = X_1, \qquad (m_1 + m_2)\, y_1'' = Y_1 + Y_2 \\ (m_1 + m_2)\, z_1'' = Z_1 + Z_2, \qquad m_2x_2'' = X_2 \end{gathered} \tag{7.2}$$

In order to find the dynamic conditions of fulfillment of prescribed constraints (7.1), we liberate the points M_1 and M_2 from these constraints. We take the coordinates y_2 and z_2 of the point M_2 as the additional generalized coordinates.

Having constructed the Lagrange equations for the coordinates y_2 and z_2 and eliminated the variables y_2 and z_2 from the resulting equations with the aid of (7.1), we arrive at the required dynamic conditions

$$m_2 y_1'' = Y_2, \qquad m_2 z_1'' = Z_2 \tag{7.3}$$

Taking account of these relations, we transform Eqs. (7.2) into

$$m_1 x_1'' = X_1, \qquad m_1 y_1'' = Y_1, \qquad m_1 z_1'' = Z_1, \qquad m_2 x_2'' = X_2$$

The significance of these equations is evident. The first three equations describe the free spatial motion of the pursued point M_1. The last equation is the equation of motion of the pursuing point along the straight homing line. Dynamic conditions (7.3) express the requirements imposed on the force components acting on the pursuing point orthogonally to the homing line.

b) Appell's problem ([1], p. 351). A plate Σ situated in a stationary horizontal plane is hinged at the point C to a round disk Σ_1 lying in the same plane and rotating about its fixed center O. The constant force F parallel to the stationary straight line Ox acts on the plate Σ at the point A lying on the straight line connecting the point C with the center of gravity G. By means of a special coupling the servomotor M acts on the disk Σ_1 in such a way as to maintain constant the relationship between angles

$$\alpha - \beta = \pi/2$$
$$\alpha = (Ox,\ Oc), \quad \beta = (Ox,\ CA), \quad OC = R, \quad CA = a, \quad CG = b \tag{7.4}$$

Thus, a controlled system has been defined. The control parameter u is the (algebraic) torque exerted by the servomotor on the disk Σ_1. We are given the prescribed constraint (7.4) which must be fulfilled during the motion of the system.

Let us write out the equation of motion of the above mechanical system assuming that the prescribed constraint is fulfilled. We add the latter to the real constraints of the system. Our mechanical system then has one degree of freedom. Let us take the angle α as its generalized coordinate.

The kinetic energy of the system in terms of α' can be written as

$$2T = [M(R^2 + b^2 + k^2) + I_1]\alpha'^2$$

Here M is the mass of the plate Σ, Mk^2 is its moment of inertia with respect to the point G, and I_1 is the moment of inertia of the disk with respect to the axis of rotation. The generalized force corresponding to the coordinate α can be written as

$$Q_\alpha = F(-R\sin\alpha + a\cos\alpha) + u$$

Constructing the Lagrange equation, we arrive at the equation of motion of the system

$$[M(R^2 + b^2 + k^2) + I_1]\alpha'' + F(R\sin\alpha - a\cos\alpha) = u \tag{7.5}$$

Next let us write out the dynamic condition of fulfillment of the prescribed constraint Liberating the system from prescribed constraint (7.4), we take the angle β as the additional generalized coordinate. The kinetic energy T of the liberated system can be written as

$$2T = M[R^2\alpha'^2 + b^2\beta'^2 + 2R\alpha'\beta'\cos(\alpha - \beta) + k^2\beta'^2] + I_1\alpha'^2$$

The generalized force corresponding to the coordinate β is given by

$$Q_\beta = -Fa\sin\beta$$

Now let us construct the Lagrange equation for the coordinate β. Eliminating β by means of Eq. (7.4), we obtain the required dynamic condition

$$M(b^2 + k^2)\alpha'' - MRb\alpha'^2 = Fa\cos\alpha$$

Let us return to the variable β (we can do this by virtue of the one-to-one correspondence between α and β set up in (7.4)). We obtain Eq.

$$M(b^2 + k^2)\beta'' - MRb\beta'^2 + Fa\sin\beta = 0 \tag{7.6}$$

which is analogous to Appell's Eq. (3) ([1], p. 351).

Appell interprets this equation as the equation of motion of the system under the servoconstraint and points out its fundamental distinctness from Eq. (4) ([1], p. 352) describing the motion of the system in the case where constraint (7.4) is effected by direct tangency between Σ and Σ_1. Appell attributes this distinctness to the special character of fulfillment of servoconstraint (7.4).

The above solution of the problem indicates, however, that Appell was in error. Eq. (7.6) is the dynamic condition of fulfillment of prescribed constraint (7.4) imposed on the system. The equation of motion of the system is Eq. (7.5), which on being stated in terms of the variable β becomes

$$[M(R^2+b^2+k^2)+I_1]\beta''+F(R\cos\beta+a\sin\beta)=u$$

The above equation is close to Appell's Eq. (4), differing from it only in its right-hand side. This is quite natural in view of the controlling action of u on the disk Σ_1.

Now let us consider some examples of nonholonomic systems.

c) **A pursuit problem.** The material point M_2 pursues another material point M_1, approaching it by the pursuit method. We are to construct the equations of the problem.

The pursuit method presupposes that the vector of the absolute velocity of the pursuing point M_2 is continuously pointed at the pursued point during the motion of the points. In other words, the condition

$$x_2'/(x_1-x_2)=y_2'/(y_1-y_2)=z_2'/(z_1-z_2) \tag{7.7}$$

must be fulfilled during the motion of the points. In this expression x_1, y_1, z_1 and x_2, y_2, z_2 denote the coordinates of the points M_1 and M_2, respectively. These relations are the prescribed constraint equations to which the motion of the points is subject. Eqs. (7.7) are clearly nonintegrable.

Let us write the equations of motion of the points M_1 and M_2 taking account of constraints (7.7). Regarding (7.7) as the real constraints of the system, we take as our velocity parameters the components x_1', y_1', z_1' of the velocity of the point M_1 and the common ratio ω in Eqs. (7.7). Eqs. (7.7) then give us

$$x_2'=\omega(x_1-x_2),\qquad y_2'=\omega(y_1-y_2),\qquad z_2'=\omega(z_1-z_2) \tag{7.8}$$

From this we have

$$\begin{aligned}x_2''&=\omega'(x_1-x_2)+\omega[x_1'-\omega(x_1-x_2)]\\ y_2''&=\omega'(y_1-y_2)+\omega[y_1'-\omega(y_1-y_2)]\\ z_2''&=\omega'(z_1-z_2)+\omega[z_1'-\omega(z_1-z_2)]\end{aligned}$$

Let us construct the energy of accelerations of the system of points M_1 and M_2,

$$2S=m_1(x_1''^2+y_1''^2+z_1''^2)+m_2(x_2''^2+y_2''^2+z_2''^2)$$

Taking account in S of the above expressions for x_2'', y_2'', z_2'', we write out the Appell equations for the velocity parameters x_1', y_1', z_1', ω,

$$\begin{gathered}m_1x_1''=X_1,\qquad m_1y_1''=Y_1,\qquad m_1z_1''=Z_1\\ m_2\{\omega'(x_1-x_2)+\omega[x_1'-\omega(x_1-x_2)]\}(x_1-x_2)+\\ +m_2\{\omega'(y_1-y_2)+\omega[y_1'-\omega(y_1-y_2)]\}(y_1-y_2)+\\ +m_2\{\omega'(z_1-z_2)+\omega[z_1'-\omega(z_1-z_2)]\}(z_1-z_2)=\\ =X_2(x_1-x_2)+Y_2(y_1-y_2)+Z_2(z_1-z_2)\end{gathered} \tag{7.9}$$

The mechanical significance of the above equations is as follows. The first three equations are clearly the equations of free spatial motion of the pursued point M_1. In order to expose the mechanical meaning of the last Eq. of system (7.9), we introduce into our discussion the velocity v of the point M_2. As is evident from (7.8), this velocity is related to ω by Eq.

$$v = \omega \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

Hence we have

$$v' = \omega' [(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2]^{1/2} +$$
$$+ \omega \{(x_1 - x_2) [x_1' - \omega (x_1 - x_2)] + (y_1 - y_2) [y_1' - \omega (y_1 - y_2)] +$$
$$+ (z_1 - z_2) [z_1' - \omega (z_1 - z_2)]\} [(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2]^{-1/2}$$

so that the last equation of system (7.9) can be written as

$$m_2 v' = \frac{X_2 (x_1 - x_2) + Y_2 (y_1 - y_2) + Z_2 (z_1 - z_2)}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}}$$

This equation describes the variation of the velocity of the pursuing point M_2.

Now let us construct the dynamic conditions of fulfillment of constraints (7.7). Rejecting the prescribed constraints, we take in place of Eqs. (7.8) equations such as

$$x_2' = \omega (x_1 - x_2), \quad y_2' = \omega (y_1 - y_2) + \sigma_1, \quad z_2' = \omega (z_1 - z_2) + \sigma_2 \tag{7.10}$$

These equations can be solved for the variables ω, σ_1, σ_2. Hence, σ_1 and σ_2 can be taken as the additional velocity parameters. The transition from Eqs. (7.10) to Eqs. (7.8) is effected by equating the variables σ_1 and σ_2 to zero. But this transition is equivalent to consideration of prescribed constraints (7.7). Hence, the equations of the prescribed constraints with the variables σ_1 and σ_2 take the form $\sigma_1 = 0$ and $\sigma_2 = 0$.

We construct the energy of accelerations of the system of points M_1 and M_2 with allowance for Expressions (7.10) and write out the Appell equations in the variables σ_1 and σ_2. Taking account of the prescribed constraints in the resulting equations, i.e. setting σ_1 and σ_2 equal to zero in these equations, we arrive at the following dynamic conditions:

$$\begin{aligned} m_2 \{\omega' (y_1 - y_2) + \omega [y_1' - \omega (y_1 - y_2)]\} &= Y_2 \\ m_2 \{\omega' (z_1 - z_2) + \omega [z_1' - \omega (z_1 - z_2)]\} &= Z_2 \end{aligned} \tag{7.11}$$

Eqs. (7.11), (7.9), and (7.8) form the complete system of equations for the pursuit problem considered.

d) **Appell's problem** ([1], p. 354). The material plane P can slide translationally over the stationary horizontal plane Oxy. The sphere Σ of radius R can roll without sliding on the plane P. The motion of the plane P is regulated automatically in such a way that the center of the sphere moves uniformly about the axis Oz with the angular velocity ω relative to the stationary axes Ox, Oy, Oz. We are required to construct the equations of the problem.

The motion of the sphere is pure rolling on the plane P. The center of the sphere must rotate uniformly about the axis Oz. The first limitation is the real constraint and the second the prescribed constraint imposed on the motion of the sphere. Let us write their equations. By ξ, η we denote the coordinates of the center of the sphere, by p, q, r the components of the instantaneous angular velocity of the sphere, and by u, v the coordinates of some point on the plane. The requirement that the center of the sphere must describe a circle can then be written in the form of two Eqs.,

$$\xi' + \omega\eta = 0, \qquad \eta' - \omega\xi = 0 \tag{7.12}$$

The condition of no sliding on the part of the sphere yields

$$\xi' - qR = u', \qquad \eta' + pR = v' \tag{7.13}$$

The motion of the sphere is controlled. Formally this is expressed by the dependence of Eqs. (7.13) on the derivatives u', v' of the coordinates of one of the points on the plane P which in this case play the role of the control parameters.

Let us construct the equations of motion of the sphere. The equations of the prescribed constraints of the system (7.12) do not depend on the control parameters. The equations of the real constraints (7.13) are parametric and are therefore allowed for by means of indefinite multipliers.

Taking into account Eqs. (7.12) of the prescribed constraints of the system, we take as our velocity parameters the components p, q, r of the angular velocity of the sphere. The energy of accelerations of the sphere can then be written as

$$2S = M\omega^2(\xi^2 + \eta^2) + {}^2/_5 MR^2(p'^2 + q'^2 + r'^2)$$

where M is the mass of the sphere. Let us construct the Appell equations in terms of the velocity parameters p, q, r. Taking account of the fact that the sphere moves by inertia so that the generalized forces are equal to zero, we arrive at the following equations of motion of the sphere:

$$2MRp' - 5\lambda_2 = 0, \qquad 2MRq' + 5\lambda_1 = 0, \qquad r' = 0 \tag{7.14}$$

where λ_1 and λ_2 are indefinite multipliers associated with the first and second Eqs. of (7.13), respectively.

To Eqs. (7.14) we must add the dynamic conditions of fulfillment of prescribed constraints (7.12). To this end we liberate the sphere from prescribed constraints (7.12) and complement the system of velocity parameters. As our additional velocity parameters we take the derivatives ξ', η'. The energy of accelerations of the liberated system is given by Expression

$$2S = M(\xi''^2 + \eta''^2) + {}^2/_5 MR^2(p'^2 + q'^2 + r'^2)$$

Constructing the Appell equations for the additional velocity parameters ξ', η' and allowing in then for the prescribed constraint equations (7.12), we arrive at the following dynamic conditions:

$$M\omega^2\xi + \lambda_1 = 0, \qquad M\omega^2\eta + \lambda_2 = 0 \tag{7.15}$$

Equations (7.12) to (7.15) form the complete system of equations of the problem. Eliminating the indefinite multipliers from Eqs. (7.14) with the aid of Eqs. (7.15), we obtain

$$2Rp' + 5\omega^2\eta = 0, \qquad 2Rq' - 5\omega^2\xi = 0, \qquad r' = 0 \tag{7.16}$$

Equations (7.16) and (7.12) describe the motion of the sphere. Eqs. (7.13) give the velocity with which the plane P must be moved in order to impart the required motion to the sphere.

The above equations differ from those given by Appell (Equations (7) of [1], p. 355). The latter can be obtained readily, however, by differentiating Eqs. (7.13) with respect to time and then applying Eqs. (7.12) and (7.16). Incidentally, this implies that Appell's equations are of higher order than ours in this case.

e) Example of a prescribed constraint which cannot be realized by a controlled mechanical system. A material ring slides freely along a smooth rod hinged at one end. The ring moves under the action of the force F which pulls it toward the free end of the rod. We are to determine the motion (rotation) of the rod about its fixed end which will oblige the ring to describe a circle with its center at the pivot point of the rod.

We conclude immediately that the required control does not exist. This is evident from the fact that the force F and the inertial force acting on the ring are both directed outside any circle with its center at the rod pivot point.

Now let us demonstrate this conclusion formally. We take the angle of rotation of the ring as the control parameter u. the directing rod along which the ring slides is a real parametric constraint. Let us find the dynamic condition of its fulfillment. With allowance for the prescribed constraint, the ring can be said to have one degree of freedom (in accordance with the general theory, we ignore the parametric real constraint).

As our generalized coordinate we take the polar angle α of the ring. Then the real constraint to which the ring is subject can be written as

$$f = \alpha - u = 0$$

As the additional generalized coordinate we take the polar distance r from the ring to the rod pivot point.

The kinetic energy T of the ring liberated from the prescribed constraint (without regard for the real parametric constraint) can be written as

$$2T = m\,(r'^2 + r^2\alpha'^2)$$

Let us construct the Lagrange equation for the variable r. Taking account of the fact that the equation of the real parametric constraint is independent of r, we obtain

$$mr' - mr\alpha'^2 = F$$

Taking account of the prescribed constraint equation (in this case it is $r = c =$ const), we arrive at the dynamic condition

$$-\,mc\alpha'^2 = F$$

Since $F > 0$, it is clear that this condition cannot be fulfilled. Q.E.D.

8. The above analysis of the motion of controlled mechanical systems with prescribed constraints would be incomplete without the following remarks.

In solving actual problems one encounters situations in which the control parameters of the mechanical system are or must be in some way inter-related. Thus, in one of the Appell problems considered here the motion of the sphere was assumed to occur by inertia. The control parameters were the components of one of the points (to be specific, let us call this point A) of the plane P. But it is quite possible for the sphere to be acted on by forces which, moreover, depend on the position of the plane P. In this case the coordinates of the plane P together with the components of the velocity of the point A must also be considered as control parameters. It is clear, however, that such a set of control parameters is not independent. The indicated parameters are, in fact, related differentially.

Another example is the motion of a controlled mechanical system with prescribed constraints in which the number of system control parameters exceeds the number of prescribed constraints to be fulfilled. The excess control can be eliminated by imposing additional restrictions. The latter can take the form of optimum control criteria. It is possible, however, that the required uniqueness of system motion might be achieved simply by interrelating the control parameters in some way.

This necessarily gives rise to the question of how given relationships between system control parameters affect the derivation of the equations of motion of a system with prescribed constraints. In answer to this we note that in deriving the equations of a given problem we in no case assumed the control parameters to be independent. Thus, insofar as such dependences do occur, they must be added to the equations of the problem obtained by the technique developed above.

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